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Sum of Some Fractional Analytic Functions

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Abstract: **In this paper, based on a new multiplication of fractional analytic functions, the sum of some fractional analytic functions is obtained. On the other hand, some examples are provided to illustrate our result. In fact, our result is a generalization of classical result.**

Keyword: **New multiplication, fractional analytic functions, sum, examples.**

I. INTRODUCTION

Fractional calculus is a mathematical analysis tool used to study arbitrary order derivatives and integrals. It unifies and extends the concepts of integer order derivatives and integrals. Generally, many scientists do not know these fractional integrals and derivatives, and they have not been used in pure mathematical context until recent years. However, in the past few decades, the fractional integrals and derivatives have frequently appeared in many scientific fields such as mechanics, viscoelasticity, physics, biology, economics, control theory, and electrical engineering [1-11].

However, the definition of fractional derivative is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, conformable fractional derivative, Jumarie's modified R-L fractional derivative [12-16]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with ordinary calculus.

In this paper, based on a new multiplication of fractional analytic functions, we find the sum of some α-fractional analytic functions:

$$\sum_{n=0}^{m} 2^n \cdot \left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2^n)}\right]^{\otimes_\alpha (-1)}, \tag{1}$$

where $0<\alpha\leq 1$, and m is a non-negative integer. In fact, our result is a generalization of traditional result of $\alpha=1$.

II. PRELIMINARIES

At First, we introduce the definition of fractional analytic function.

Definition 2.1 ([17])**:** If x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, a new multiplication of fractional analytic functions is introduced below.

Definition 2.2 ([18])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{2}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}. \tag{3}$$

Then we define

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\\&=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}.\end{aligned} \tag{4}$$

Equivalently,

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}.\end{aligned} \tag{5}$$

Definition 2.3 ([19])**:** If $0<\alpha\le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{6}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \tag{7}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha(g_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{a_n}{n!}(g_\alpha(x^\alpha))^{\otimes_\alpha n}, \tag{8}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha(f_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{b_n}{n!}(f_\alpha(x^\alpha))^{\otimes_\alpha n}. \tag{9}$$

Definition 2.4([20])**:** Let $0<\alpha\le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $(f_\alpha(x^\alpha))^{\otimes_\alpha n}=f_\alpha(x^\alpha)\otimes_\alpha\cdots\otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)=1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $(f_\alpha(x^\alpha))^{\otimes_\alpha -1}$.

III. MAIN RESULT

In this section, the sum of some fractional analytic functions is obtained, and we give some examples to illustrate our result.

Theorem 3.1: *If* $0<\alpha\le 1$, *and* m *is a non-negative integer. Then*

$$\begin{aligned}&\sum_{n=0}^{m}2^n\cdot\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2^n)}\right]^{\otimes_\alpha(-1)}\\&=\left[(2^{m+1}-1)-2^{m+1}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2^{m+1})}\right]\\&\otimes\left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2^{m+1})}+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2^{m+1}+1)}\right]^{\otimes_\alpha(-1)}.\end{aligned} \tag{10}$$

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Proof $\sum_{n=0}^{m} 2^n \cdot \left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2^n)}\right]^{\otimes_\alpha (-1)}$

$$= \left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (1)}\right]^{\otimes_\alpha (-1)} + 2\cdot\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2)}\right]^{\otimes_\alpha (-1)} + 4\cdot\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (4)}\right]^{\otimes_\alpha (-1)} +$$

$$\cdots + 2^m\cdot\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2^m)}\right]^{\otimes_\alpha (-1)}$$

$$= \left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (1)}\right]^{\otimes_\alpha (-1)} + \left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (1)}\right]^{\otimes_\alpha (-1)} + 2\cdot\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2)}\right]^{\otimes_\alpha (-1)}$$

$$+4\cdot\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (4)}\right]^{\otimes_\alpha (-1)} + \cdots + 2^m\cdot\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2^m)}\right]^{\otimes_\alpha (-1)} - \left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (1)}\right]^{\otimes_\alpha (-1)}$$

$$= 2\cdot\left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2)}\right]^{\otimes_\alpha (-1)} + 2\cdot\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2)}\right]^{\otimes_\alpha (-1)}$$

$$+4\cdot\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (4)}\right]^{\otimes_\alpha (-1)} + \cdots + 2^m\cdot\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2^m)}\right]^{\otimes_\alpha (-1)} - \left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (1)}\right]^{\otimes_\alpha (-1)}$$

$$= 4\cdot\left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (4)}\right]^{\otimes_\alpha (-1)} + 4\cdot\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (4)}\right]^{\otimes_\alpha (-1)} + \cdots$$

$$+2^m\cdot\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2^m)}\right]^{\otimes_\alpha (-1)} - \left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (1)}\right]^{\otimes_\alpha (-1)}$$

$$= 2^m\cdot\left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2^m)}\right]^{\otimes_\alpha (-1)} + 2^m\cdot\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2^m)}\right]^{\otimes_\alpha (-1)} - \left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (1)}\right]^{\otimes_\alpha (-1)}$$

$$= 2^{m+1}\cdot\left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2^{m+1})}\right]^{\otimes_\alpha (-1)} - \left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (1)}\right]^{\otimes_\alpha (-1)}$$

$$= \left[(2^{m+1}-1) - 2^{m+1}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right) + \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2^{m+1})}\right]$$

$$\otimes\left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right) - \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2^{m+1})} + \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2^{m+1}+1)}\right]^{\otimes_\alpha (-1)}.$$

Q.e.d

Example 3.2: Let $0 < \alpha \leq 1$, then

$$\sum_{n=0}^{6} 2^n \cdot \left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2^n)}\right]^{\otimes_\alpha (-1)}$$

$$= \left[127 - 128\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right) + \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (128)}\right] \otimes \left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right) - \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (128)} + \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (129)}\right]^{\otimes_\alpha (-1)}. \quad (11)$$

And

$$\sum_{n=0}^{10} 2^n \cdot \left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2^n)}\right]^{\otimes_\alpha (-1)}$$

$$=\left[2047-2048\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2048)}\right]\otimes\left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2048)}+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2049)}\right]^{\otimes_\alpha (-1)}. \quad (12)$$

IV. CONCLUSION

In this paper, a sum of some fractional analytic functions is obtained based on a new multiplication of fractional analytic functions. Moreover, some examples are given to illustrate our result. In fact, our result is a generalization of traditional result. In the future, we will continue to study the problems in engineering mathematics and fractional differential equations.

REFERENCES

[1] J. H. He, Some applications of nonlinear fractional differential equations and their approximations, Science Technology and Society, vol. 15, pp. 86-90, 1999.

[2] W. S. Chung, Fractional Newton mechanics with conformable fractional derivative, Journal of Computational and Applied Mathematics, vol. 290, pp. 150-158, 2015.

[3] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[4] E. Soczkiewicz, Application of fractional calculus in the theory of viscoelasticity, Molecular and Quantum Acoustics vol.23, pp. 397-404, 2002.

[5] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, Fractals and Fractional Calculus in Continuum Mechanics, pp. 291-348, Springer, Wien, Germany, 1997.

[6] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, vol. 8, no. 5, 660, 2020.

[7] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp, 41-45, 2016.

[8] R. Magin, Fractional calculus in bioengineering, part 1, Critical Reviews in Biomedical Engineering, vol. 32, no,1. pp.1-104, 2004.

[9] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, Vol. 7, Issue 8, pp. 3422-3425, 2020.

[10] G. Jumarie, Path probability of random fractional systems defined by white noises in coarse-grained time applications of fractional entropy, Fractional Differential Equations, vol. 1, pp. 45-87, 2011.

[11] T. M. Atanacković, S. Pilipović, B. Stanković, and D. Zorica, Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes, Mechanical Engineering and Solid Mechanics, Wiley-ISTE, Croydon (2014).

[12] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, Calif, USA, 1999.

[13] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, Inc., 1974.

[14] K. S. Miller and B. Ross, An introduction to the Fractional Calculus and Fractional Differential Equations, A Wiley-Interscience Publication, John Wiley & Sons, New York, USA, 1993.

[15] S. Das, Functional Fractional Calculus for System Identification and Control, 2nd ed., Springer-Verlag, Berlin, 2011.

[16] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, 2010.

[17] C. -H. Yu, Study of fractional analytic functions and local fractional calculus, International Journal of Scientific Research in Science, Engineering and Technology, vol. 8, no. 5, pp. 39-46, 2021.

[18] C. -H. Yu, Solutions of a fractional algebraic equation, International Journal of Novel Research in Engineering and Science, vol. 9, no. 2, pp. 40-43, 2023.

[19] C. -H. Yu, Exact solutions of some fractional power series, International Journal of Engineering Research and Reviews, vol. 11, no. 1, pp. 36-40, 2023.

[20] C. -H. Yu, Infinite series expressions for the values of some fractional analytic functions, International Journal of Interdisciplinary Research and Innovations, vol. 11, no. 1, pp. 80-85, 2023.